ANALYTICAL SOLUTIONS OF THE SCHRODINGER EQUATION. GROUND STATE ENERGIES AND WAVE FUNCTIONS

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Dedicated to Professor Rudolf Zahradnik on the occasion of his 70th birthday.

It is shown that there are two generalizations of some well-known analytically solvable problems leading to exact analytical solutions of the Schrodinger equation for the ground state and a few low lying excited states. In this paper, the ground state energies and wave functions are discussed.
Key words: Schrodinger equation; Analytic solutions; Ground state; Quantum chemistry.

In this paper, we investigate the bound states of the one-dimensional Schrodinger equation

$$H\psi(x) = E\psi(x) \tag{1}$$

with the Hamiltonian

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \ . \tag{2}$$

Examples of potentials for which is the Schrodinger equation (*1*) analytically solvable include the harmonic potential, some anharmonic potentials[1–11], the Morse potential[10,12,13], the Kratzer potential[13,14,15], the Rosen–Morse potential[16], the Poschl–Teller potential[17], the Eckart potential[18], the Hulthen potential[19], the Manning–Rosen potential[20] and some other cases (see *e.g.* refs[8,13]). In this paper, we investigate two generali-

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zations of these potentials for which the Schrodinger equation (*1*) can be solved analytically. For the sake of simplicity, we investigate the ground state only.

In our earlier paper[10] we used the expansion of the wave function in the form

$$\psi = \sum_i c_i \psi_i \ , \tag{3}$$

where

$$\psi_i(x) = [f(x)]^i \, g(x) \ . \tag{4}$$

Here, $f(x) = f$ and $g(x) = g$ are functions which are determined from the condition that analytical solutions of the Schrödinger equation exist. There is a chance of finding analytical solutions of the Schrödinger equation if the Hamiltonian H transforms the set of the basis functions ψ_i into itself

$$H\psi_i = \sum_j h_{ij} \psi_j \ , \tag{5}$$

where h_{ij} are numerical coefficients. Assuming linear independence of the functions ψ_i, we get from Eqs (*1*)–(*5*) a non-hermitian eigenvalue problem

$$\sum_i c_i h_{ij} = E c_j \ . \tag{6}$$

In general case, when the left eigenvectors $c = \{c_i\}$ have the infinite number of non-zero components, the solution of the problem (*6*) is difficult. On the other hand, if the eigenvector c has only a finite number of non-zero components, Eq. (*6*) can be reduced to a finite order problem and there is a chance to find analytical solutions.

Sufficient conditions which guarantee the property (*5*) can be written in the form[10]

$$\frac{df(x)}{dx} = \sum_i f_i \, [f(x)]^i \ , \tag{7}$$

$$\frac{dg(x)}{dx} = -g(x) \sum_i g_i \, [f(x)]^i \ , \tag{8}$$

and

$$V(x) = \sum_i V_i\,[f(x)]^i \;, \tag{9}$$

where f_i, g_i and V_i are numerical coefficients. The function $f(x)$ appearing in Eq. (*9*) can be understood as a variable transformation from the anharmonic potential in the variable x to the potential in a new variable $f(x)$. If the coefficients f_i and g_i are known, the functions f and g can be obtained by inverting the function[10]

$$x(f) = \int \frac{1}{\sum_i f_i\, f^{\,i}}\, \mathrm{d}f \tag{10}$$

and calculating

$$g(x) = \exp\left(-\int \sum_i g_i\, f(x)^i\, \mathrm{d}x\right) \;. \tag{11}$$

The matrix ***h*** appearing in Eq. (*6*) has the form[10]

$$h_{m,m+i} = -m(m-1)\sum_j f_j\, f_{i-j+2} + m\sum_j\,(2f_j\, g_{i-j+1} - j\, f_j\, f_{i-j+2}) + \\ +\sum_j\,(j\, f_{i-j+1}\, g_j - g_j\, g_{i-j}) + V_i \;. \tag{12}$$

In this formulation, the problem of finding analytical solutions of the Schrodinger equation (*1*) leads to the question when the eigenvalue problem (*6*) can be reduced to the problem of a finite order. For very low orders, we may be able to solve this problem explicitly and find exact analytical solutions. For large orders, a numerical solution of the eigenvalue problem may be necessary.

Significant advantage of our algebraic approach is its generality. We do not assume any concrete form of the function $f(x)$ nor the values of the potential coefficients V_i. Changing the values of the coefficients f_i, we can change the form of the function $f(x)$. Further, changing the coefficients V_i, we can change the form of the potential $V(x)$ for a given function $f(x)$. We see that such potentials are rather general and can be better adapted to experimental potentials than in the usual approaches.

First we summarize the standard approach, in which the function $f(x)$ obeyes the equation $\mathrm{d}f/\mathrm{d}x = f_0 + f_1 f + f_2 f^2$ and the potential $V(x)$ is quadratic in the function f, $V = V_0 + V_1 f + V_2 f^2$ (section Standard Approach). Then, we investigate generalization in which a more general potential $V = V_0 + V_1 f + V_2 f^2 + ... + V_{2M} f^{2M}$ with the same class of functions $f(x)$ as above is assumed (section First Generalization). In the following section Second Generalization, we investigate another generalization in which more general functions $f(x)$ obeying the equation $\mathrm{d}f/\mathrm{d}x = f_0 + f_1 f + ... + f_N f^N$ are considered. The solution of the problem (*6*) is discussed in section Eigenvalue Problem. The ground state energies and wave functions for general case are investigated in section Ground State.

STANDARD APPROACH

The harmonic oscillator, the Morse, Kratzer, Rosen–Morse, Poschl–Teller, Eckart, Hulthen and Manning–Rosen oscillators[12,14,16–20] are based on the equations $\mathrm{d}f/\mathrm{d}x = f_0 + f_1 f + f_2 f^2$ and $V = V_0 + V_1 f + V_2 f^2$, where $V_2 > 0$. As is was shown previously[10], the number of potential constraints (conditions on the potentials coefficients V_i) equals $M - 1$. Since $M = 1$ in this case, there are no potential constraints for such potentials and all analytic solutions belong to the same potential. In the examples given below, physically insignificant integration constants are omitted.

Harmonic oscillator. The most simple form of Eq. (*7*) is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 1 \quad . \tag{13}$$

This equation with $f_0 = 1$ leads to

$$f = x \quad . \tag{14}$$

It gives the parabolic potential $V = V_0 + V_1 x + V_2 x^2$. For $V_0 = V_1 = 0$ and $V_2 = 1$, the harmonic oscillator is obtained.

Morse potential.

$$\frac{\mathrm{d}f}{\mathrm{d}x} = a - f \tag{15}$$

For $f_0 = a$ and $f_1 = -1$, this equation gives

$$f = a - \exp(-x) \quad . \tag{16}$$

For $a = 1$, the potential $V = V_0 + V_1 f + V_2 f^2$ is equivalent to the well-known Morse potential[10,12,13] $V = D[1 - \exp(-\alpha(x - x_0)/a)]^2$.

Kratzer potential.

$$\frac{\mathrm{d}f}{\mathrm{d}x} = (a - f)^2 \qquad (17)$$

For $f_0 = a^2$, $f_1 = -2a$ and $f_2 = 1$, this equation gives

$$f = a - 1/x \ . \qquad (18)$$

For $f = 1 - 1/x$, the potential $V = V_0 + V_1 f + V_2 f^2$ is equivalent to the Kratzer potential[13–15] $V = -2D[a/x - (a/x)^2/2]$. For $f = -1/x$, the one-dimensional Coulomb potential $V = V_2 f$ is obtained.

Symmetric Rosen–Morse and Poschl–Teller potentials.

$$\frac{\mathrm{d}f}{\mathrm{d}x} = a - f^2 \qquad (19)$$

For $f_0 = a$ and $f_2 = -1$, this equation yields

$$f = \sqrt{a} \tanh(\sqrt{a}x) \ . \qquad (20)$$

For $a = 1$, the symmetric Rosen–Morse potential $V = V_2 \tanh(x)^2$ is obtained[16]. For $a = -1$, we get the symmetric Poschl–Teller potential[17] $V = V_2 \tan(x)^2$. This potential can also be obtained from the equation $\mathrm{d}f/\mathrm{d}x = \mathrm{i}(1 - f^2)$ leading to $f = \mathrm{i} \tan(x)$.

Some other known potentials. A more general equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = k(a + f)(b - f) \qquad (21)$$

with $f_0 = kab$, $f_1 = -k(a - b)$ and $f_2 = -k$ leads to

$$f = [b \pm a \exp(-k(a + b)x)]/[1 \mp \exp(-k(a + b)x)] \ . \qquad (22)$$

Similarly, the equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = k(a - f)(b - f) \qquad (23)$$

with $f_0 = kab$, $f_1 = -k(a + b)$ and $f_2 = k$ gives

$$f = [b \pm a \exp(-k(a-b)x)]/[1 \pm \exp(-k(a-b)x)] \quad . \qquad (24)$$

Depending on the values of a, b, k and V_0, V_1, V_2, we can get the following potentials:

(i) The Rosen–Morse potential[16] $V = A \tanh(x) - B/\cosh(x)^2$ for Eq. (*21*), $a = -1$, $b = 0$, $k = 2$, $V_0 = -A$, $V_1 = 2A - 4B$ and $V_2 = 4B$.

(ii) The Eckart potential[18] $V = -A\xi/(1-\xi) - B\xi/(1-\xi)^2$, where $\xi = -\exp(x)$ for Eq. (*23*), $a = -1$, $b = 0$, $k = 1$, $V_0 = 0$, $V_1 = -A - B$ and $V_2 = -B$.

(iii) The Hulthen potential[13,19] $V = -A\xi/(1-\xi)$, where $\xi = \exp(-x)$ for Eq. (*21*), $a = 1$, $b = 0$, $k = 1$, $V_0 = 0$, $V_1 = -A$ and $V_2 = 0$.

(iv) The Manning–Rosen potential[20] $V = -A\xi/(1-\xi) + B\xi^2/(1-\xi)^2$, where $\xi = \exp(-x)$ for Eq. (*21*), $a = -1$, $b = 0$, $k = -1$, $V_0 = 0$, $V_1 = -A$ and $V_2 = B$.

FIRST GENERALIZATION

The most simple way of generalization of the examples given above is to take functions $f(x)$ given in section Standard Approach and consider higher order potentials in $f(x)$ ($M \geq 2$). It has one important consequence. According to ref.[10], the eigenvalue problem (*6*) is analytically solvable only if $M - 1$ potential constraints on the potential coefficients V_i are introduced. Therefore, the analytical solution is not possible for all the potential coefficients as in the standard case. Even more significant limitation is that the potential constraints depend usually on the number of the functions ψ_i in the linear combination (*3*). This means, that different states (the ground state, the first excited state, *etc.*) belong usually to different potential coefficients, *i.e.* to different potentials. A few examples is given below.

Anharmonic oscillators. For $M > 1$, Eq. (*13*) gives the potentials of the anharmonic oscillators $V = V_0 + V_1 x + ... + V_{2M} x^{2M}$. Examples of the analytic solutions for the anharmonic oscillators can be found for example in ref.[10]. The oscillators with the potential depending on $|x|$ can also be investigated[11].

Generalized Morse potential. For $M > 1$, Eq. (*15*) leads to the generalized Morse potential $V = V_0 + V_1 f + V_2 f^2 + ... + V_{2M} f^{2M}$, where $f = 1 - \exp(-x)$. For examples of analytical solutions, see ref.[10].

Generalized Kratzer potential. For $M > 1$, Eq. (*17*) yields the generalized Kratzer potential $V = V_0 + V_1 f + V_2 f^2 + ... + V_{2M} f^{2M}$, where $f = 1 - 1/x$. For examples of analytical solutions, see ref.[15].

Generalized symmetric Rosen–Morse potential. According to our knowledge, the generalized symmetric Rosen–Morse potential $V = V_0 + V_1 f + V_2 f^2 + ... + V_{2M} f^{2M}$ with the function f given by Eq. (*20*) has not been investigated till now.

Generalization of some other known potentials. According to our knowledge, the generalized Rosen–Morse, Eckart, Hulthen and Mannig–Rosen potentials $V = V_0 + V_1 f + V_2 f^2 + ... + V_{2M} f^{2M}$ with the function f given by Eqs (*22*) or (*24*) have not been investigated till now.

SECOND GENERALIZATION

In this section, we consider a few examples of more general equations for the function $f(x)$. We restrict ourselves to a few examples which can be of physical interest.

Potentials with fractional powers. The equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = (-1)^{k+1}(a - f)^k \ , \quad k = 2, 3, \ldots \tag{25}$$

leads to the solution

$$f = a \pm 1/[(k-1)x]^{1/(k-1)} \ . \tag{26}$$

Similarly, the equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 1/f^{\,k} \ , \quad k = 2, 3, \ldots \tag{27}$$

has the solution

$$f = (-1)^k \, [(k+1)x]^{1/(k+1)} \ . \tag{28}$$

Potentials with the fractional powers of x were discussed also in ref.[6].

Some more general potentials. The equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 1 + f^{\,N} \tag{29}$$

has the following implicit solution for N even[21]

$$x = -\frac{2}{N}\sum_{k=0}^{N/2-1} P_k \cos\frac{2k+1}{N}\pi + \frac{2}{N}\sum_{k=0}^{N/2-1} Q_k \sin\frac{2k+1}{N}\pi \tag{30}$$

and for N odd

$$x = \frac{1}{N}\ln(1+f) - \frac{2}{N}\sum_{k=0}^{(N-3)/2} P_k \cos\frac{2k+1}{N}\pi + \frac{2}{N}\sum_{k=0}^{(N-3)/2} Q_k \sin\frac{2k+1}{N}\pi \ . \qquad (31)$$

Here,

$$P_k = \frac{1}{2}\ln\left(f^2 - 2f\cos\frac{2k+1}{N}\pi + 1\right) \qquad (32)$$

and

$$Q_k = \arctan\frac{f - \cos\frac{2k+1}{N}\pi}{\sin\frac{2k+1}{N}\pi} \ . \qquad (33)$$

The solution of the equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 1 - f^N \qquad (34)$$

is analogous to the previous case and will not be given here (see ref.[21]).

It is obvious from these examples that there is a rather broad class of functions f and related potentials V given by Eqs (*7*) and (*9*) which can yield analytic solutions. Some special cases have been discussed above. Now, we discuss the solution of the eigenvalue problem (*6*).

EIGENVALUE PROBLEM

Henceforth, we assume that the potential equals

$$V(x) = \sum_{i=0}^{2M} V_i\,[f(x)]^i \ , \quad V_{2M} > 0 \ , \quad M = 1, 2, \ldots \ , \qquad (35)$$

where the function $f(x)$ is the solution of the equation

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \sum_{i=0}^{M+1} f_i\,[f(x)]^i \ . \qquad (36)$$

We assume that the coefficients V_1, ..., V_{2M}, f_0, ..., f_{M+1} and the function $f(x)$ necessary for the definition of the potential $V(x)$ are given. All the coefficients V_i and f_i and the function $f(x)$ are assumed to be real. The wave function is assumed in the form of a finite linear combination

$$\psi = \sum_{i=0}^{n} c_i \psi_i \ , \tag{37}$$

where

$$\psi_i(x) = f(x)^i \, g(x) \ . \tag{38}$$

Further we assume that the function $g(x)$ obeys the equation

$$\frac{\mathrm{d}g(x)}{\mathrm{d}x} = -g(x) \sum_{i=0}^{M} g_i \, [f(x)]^i \ , \tag{39}$$

where g_0, ... g_M are real coefficients. This equation yields

$$g(x) = \exp\left(-g_0 \, x - \sum_{i=1}^{M} g_i \int f(x)^i \, \mathrm{d}x\right) \ . \tag{40}$$

To find the solution of the Schrodinger equation (*1*) it is necessary to determine the coefficients g_i and c_i and the energy E for which is the eigenvalue problem (*6*) obeyed.

For our assumptions, the matrix (*5*) equals

$$h_{m,m+i} = -m(m-1) \sum_{j=0}^{M+1} f_j \, f_{i-j+2} + m \sum_{j=0}^{M+1} (2 f_j \, g_{i-j+1} - j \, f_j \, f_{i-j+2}) +$$
$$+ \sum_{j=0}^{M+1} (j \, f_{i-j+1} \, g_j - g_j \, g_{i-j}) + V_i \ , \quad i = -2, \ldots, 2M \ , \tag{41}$$

where the terms in the summations have the form of discrete convolutions. Here, we assume that $g_i = 0$ for $i < 0$ and $i > M$. Depending on the values of the coefficients f_i and g_i many terms in this equation equal zero.

Because of the form of these summations, the coefficients f_i and g_i contribute to two lower off-diagonals and to $2M$ upper off-diagonals of $\boldsymbol{h}$ containing V_1, ..., V_{2M}. This structure of $\boldsymbol{h}$ makes possible to reduce the infinite order problem (*6*) to a finite order one

$$\sum_{i=0}^{n} c_i h_{ij} = E c_j \ , \quad j = 0, \ldots, n \ , \qquad (42)$$

if some additional conditions are satisfied. If these conditions are obeyed, the eigenvalue problem (*42*) can be solved for small n analytically. For large n, it can be solved numerically. Following ref.[8], solutions in the latter case can be denoted as quasi-exact.

GROUND STATE

In this section, we perform general discussion of the ground state energy and wave function (n = 0 in Eq. (*42*)).

For $n = 0$, we can put $c_i = \delta_{i0}$. Then, the problem (*6*) has the $1 \times 2M$ form

$$h_{0i} = E\delta_{0i} \ , \quad i = 0, \ldots, 2M \ . \qquad (43)$$

The wave function equals

$$\psi(x) = g(x) = \exp\left(-g_0 x - \sum_{i=1}^{M} g_i \int f(x)^i \, \mathrm{d}x\right) \ . \qquad (44)$$

Since the coefficients g_i and the function f are real this function has no nodes. Therefore, if the function ψ is quadratically integrable it gives the ground state wave function with the energy $E = h_{00}$.

Because of the form of the matrix $\boldsymbol{h}$, this problem has a non-trivial solution if h_{0i} = 0, i = 1, ..., $2M$ or

$$\sum_{j=0}^{M+1} g_j \, g_{i-j} - \sum_{j=1}^{M+1} j \, f_{i-j+1} \, g_j = V_i \ , \quad i = 1, \ldots, 2M \ . \qquad (45)$$

Again, we assume here that $g_i = 0$ for $i < 0$ and $i > M$. The corresponding energy equals

$$E = h_{00} = V_0 + f_0 \, g_1 - g_0^2 \ . \qquad (46)$$

In more detail, Eq. (*45*) can be written as

$$2g_0 \, g_1 - f_1 \, g_1 - 2f_0 \, g_2 = V_1 \ , \qquad (47)$$

$$\sum_{j=0}^{i} g_j\, g_{i-j} - \sum_{j=1}^{i+1} j\, f_{i-j+1}\, g_j = V_i \ , \quad i = 2, \ldots, 2M-1 \ , \tag{48}$$

$$g_M^2 - M f_{M+1}\, g_M = V_{2M} \ . \tag{49}$$

It is obvious that this system of equations can be solved recurrently starting from the last equation (*49*).

"Quadratic" potentials M = 1. First we discuss the "quadratic" potentials

$$V = V_0 + V_1 f + V_2 f^2 \ , \quad V_2 > 0 \ . \tag{50}$$

These potentials include the examples discussed in section Standard Approach as special cases. For "quadratic" potentials, only the coefficients g_0, g_1, f_0, f_1, f_2, V_0, V_1 and V_2 can be different from zero.

The function $f(x)$ obeys the equation

$$\mathrm{d}f/\mathrm{d}x = f_0 + f_1 f + f_2 f^2 \ . \tag{51}$$

For general f_0, f_1 and f_2, this equation has the solution in the form

$$f(x) = [a \tan(ax/2) - f_1]/(2f_2) \ , \tag{52}$$

where $a = (4f_0 f_2 - f_1^2)^{1/2}$.

In a special case $f_2 = -f_0 - f_1$, we get from Eq. (*51*)

$$f(x) = (f_0 + \xi)/(f_2 + \xi) \ , \tag{53}$$

where $\xi = \exp[(2f_0 + f_1)x]$.

The wave function ψ has the form

$$\psi(x) = g(x) = \exp\left(-g_0 x - g_1 \int f(x)\, \mathrm{d}x\right) \ . \tag{54}$$

The coefficients g_i follow from Eqs (*47*) and (*49*)

$$g_1 = f_2/2 \pm (f_2^2/4 + V_2)^{1/2} \tag{55}$$

and

$$g_0 = f_1/2 + V_1/(2g_1) \ . \tag{56}$$

The energy (*46*) can be written in the form

$$E = V_0 + V_1 + V_2 - (g_0 + g_1)^2 + (f_0 + f_1 + f_2)g_1 \ . \tag{57}$$

We see that the calculation of the function $g(x)$ and the energy E for the “quadratic” potentials is straightforward. No potential constraints have to be introduced. Because of the ± sign in Eq. (*55*) there are two possible sets of the coefficients g_i. Only those leading to quadratically integrable wave functions are of interest.

Examples of the solutions for the ground state of the “quadratic” potentials can be found in the literature[10,11,15].

“Quartic” potentials M = 2. Now, we discuss the “quartic” potentials

$$V = V_0 + V_1 f + V_2 f^2 + V_3 f^3 + V_4 f^4 \ , \quad V_4 > 0 \ . \tag{58}$$

For “quartic” potentials, only the coefficients g_0, ..., g_2, f_0, ..., f_3 and V_0, ..., V_4 can be different from zero. The function $f(x)$ obeys the equation

$$\mathrm{d}f/\mathrm{d}x = f_0 + f_1 f + f_2 f^2 + f_3 f^3 \ . \tag{59}$$

The wave function ψ has the form

$$\psi(x) = g(x) = \exp\left(-g_0 x - \sum_{i=1}^{2} g_i \int f(x)^i \,\mathrm{d}x\right) \ . \tag{60}$$

The coefficients g_i follow from Eqs (*47*)–(*49*)

$$g_2 = f_3 \pm (f_3^2 + V_4)^{1/2} \ , \tag{61}$$

$$g_1 = (2f_2 g_2 + V_3)/(2g_2 - f_3) \tag{62}$$

and

$$g_0 = (2f_1 g_2 + f_2 g_1 - g_1^2 + V_2)/(2g_2) \ . \tag{63}$$

One potential constraint equals

$$V_1 = -2f_0 g_2 - f_1 g_1 + 2g_0 g_1 \ . \tag{64}$$

The energy (*46*) can be written in the form

$$E = \sum_{i=0}^{4} V_i - \left(\sum_{i=0}^{2} g_i \right)^2 + \sum_{i=0}^{3} f_i \sum_{j=1}^{2} j \, g_j \ . \tag{65}$$

Similarly to the "quadratic" potentials, the calculation of the function $g(x)$ and energy E for the "quartic" potentials with one potential constraint for the coefficient V_1 is straightforward.

We note that the boundary conditions for the wave function (*60*) cannot be obeyed in all cases. This is for example the case of the quartic oscillator with $V = x^4$, where the wave function (60) contains the term exp $(-g_2 x^3/3)$ which diverges for $x \to -\infty$ or $x \to \infty$ (ref.[10]). However, the boundary conditions can be obeyed if the potential (*58*) depending on $|x|$ instead of x is taken[11].

"Sextic" and higher order potentials M > 2. First we discuss the sextic potentials of the form

$$V = \sum_{i=0}^{6} V_i f^i \ , \qquad V_6 > 0 \ . \tag{66}$$

For "sextic" potentials, only the coefficients g_0, ..., g_3 and f_0, ..., f_4 can be different from zero. The function $f(x)$ obeys the equation

$$\mathrm{d}f/\mathrm{d}x = \sum_{i=0}^{4} f_i f^i \ . \qquad (67)$$

The wave function ψ has the form

$$\psi(x) = g(x) = \exp\,(-g_0 x - \sum_{i=1}^{3} g_i \int f(x)^i \,\mathrm{d}x) \ . \qquad (68)$$

The coefficients g_i follow from Eqs (*47*)–(*49*)

$$g_3 = 3f_4/2 \pm [(3f_4/2)^2 + V_6]^{1/2} \ , \qquad (69)$$

$$g_2 = (3f_3 g_3 + V_5)/(2g_3 - 2f_4) \ , \qquad (70)$$

$$g_1 = (3f_2 g_3 + 2f_3 g_2 - g_2^2 + V_4)/(2g_3 - f_4) \qquad (71)$$

and

$$g_0 = (3f_1 g_3 + 2f_2 g_2 + f_3 g_1 - 2g_1 g_2 + V_3)/(2g_3) \ . \qquad (72)$$

The first potential constraint is the same as for the "quartic" potentials (*64*). The second one reads

$$V_2 = -3f_0 g_3 - 2f_1 g_2 - f_2 g_1 + 2g_0 g_2 + g_1^2 \ . \qquad (73)$$

The energy (*46*) can be written in the form

$$E = \sum_{i=0}^{6} V_i - \left(\sum_{i=0}^{3} g_i \right)^2 + \sum_{i=0}^{4} f_i \sum_{i=1}^{3} i\, g_i \ . \qquad (74)$$

If the function (*68*) is quadratically integrable it describes the ground state wave function with the energy (*74*).

We note that, in constrast to the quartic anharmonic oscillator, the ground state wave function (*68*) of the sextic anharmonic oscillator with $V = x^6$ contains the term exp $(-g_3 x^4/4)$ which makes possible to obey the boundary conditions[10] $\lim_{x\to\pm\infty} \psi(x) \to 0$ if $g_3 > 0$.

Results for the higher order oscillators are analogous to those derived above and will not be given here. We can see that introducing $M - 1$ potential constraints for the potential coefficients $V_1, ..., V_{M-1}$ the function $g(x)$ and energy E can be easily investigated for an arbitrary potential given by Eq. (*35*).

We note that the boundary conditions can be obeyed for all anharmonic oscillators with $V = x^{2M}$, where $2M = 4n + 2$, $n = 1,2,....$ For $2M = 4n$, the boundary conditions can be obeyed only if $|x|$ is used in the potential instead of x.

CONCLUSIONS

In this paper, we have shown that there are two generalizations of some well-known potentials for which is the Schrodinger equation analytically solvable. The standard approaches are based on the potential $V = V_0 + V_1 f(x) + V_2 f(x)^2$, where $f(x)$ is a conveniently chosen function. The first generalization uses the same functions as in standard approaches, however, the potential includes higher order terms in the potential (*35*). The second generalization is based on more general functions $f(x)$ which must obey Eq. (*36*). It appears that the analytical solution is possible only if $M - 1$ potential constraints on the potential coefficients V_i are introduced. If the "quadratic" potentials $V = V_0 + V_1 f(x) + V_2 f(x)^2$ are considered, no additional constraints have to be introduced. In this paper, we discussed the ground state energies and wave functions. Excited states will be discussed elsewhere.

Summarizing, we have shown that some well-known potentials for which is the Schrodinger equation analytically solvable can be generalized in such a way that – at least for the ground state and a few low lying excited states – the analytical solution is still possible.

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REFERENCES

1. Flessas G. P.: *Phys. Lett. A* **1979**, *72*, 289.
2. Flessas G. P., Das K. P.: *Phys. Lett. A* **1980**, *78*, 19.
3. Magyari E.: *Phys. Lett. A* **1981**, *81*, 116.
4. Killingbeck J., Jones M. N., Thompson M. J.: *J. Phys. A: Math. Gen.* **1985**, *18*, 793.
5. Turbiner A. V., Ushveridze A. G.: *Phys. Lett. A* **1987**, *126*, 181.
6. Znojil M. in: *Rigorous Results in Quantum Mechanics* (J. Dittrich and P. Exner, Eds). World Scientific Publishing, Singapore 1991.
7. Znojil M.: *Phys. Lett. A* **1992**, *169*, 415; and references therein.

8. Ushveridze A. G.: *Quasi-Exactly Solvable Models in Quantum Mechanics*. Institute of Physics Publishing, Bristol 1994.
9. Vanden Berghe G., De Meyer H., Van Daele M: *Int. J. Theor. Phys*. **1995**, *34*, 1931.
10. Skala L., Cizek J., Dvorak J., Spirko V.: *Phys. Rev. A: At., Mol., Opt. Phys*. **1996**, *53*, 2009.
11. Skala L., Dvorak J., Kapsa V.: *Int. J. Theor. Phys*. **1998**, *36*, 2953.
12. Morse P. M.: *Phys. Rev*. **1929**, *34*, 57.
13. Flugge S.: *Rechenmethoden der Quantentheorie dargestelt in Aufgaben und Losungen*. Springer, Berlin 1947.
14. Kratzer A.: *Z. Phys*. **1920**, *3*, 289.
15. Dvorak J., Skala L.: *Bull. Soc. Sci. Lett. (Lodz)* **1996**, *46*, 11.
16. Rosen N., Morse P. M.: *Phys. Rev*. **1932**, *42*, 210.
17. Poschl G., Teller E.: *Z. Phys*. **1933**, *83*, 1439.
18. Eckart C.: *Phys. Rev*. **1930**, *35*, 1303.
19. a) Hulthen L.: *Arkiv. Math. Astron. Fys. A* **1942**, *28*, 5; b) Hulthen L.: *Arkiv. Math. Astron. Fys. B* **1942**, *29*, 1.
20. Manning M. F., Rosen N.: *Phys. Rev*. **1933**, *44*, 953.
21. Gradstein I. S., Ryshik I. M.: *Summen-, Produkt- und Integral-Tafeln*. Harri Deutsch, Thun 1981.